Functional limit theorems for generalized variations of the fractional Brownian sheet

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Introduction

- We are interested in the asymptotic behavior of non-linear functionals of high-frequency observations of strongly autocorrelated Gaussian random fields.
- By random fields, we mean stochastic processes with multiple parameters.
- In particular, we study the "phase transition" from Gaussian limits (central limit theorems) to non-Gaussian limits.
- We are also interested in the qualitative differences between the Gaussian and non-Gaussian limits (aside from Gaussianity/non-Gaussianity).

Review of the one-parameter case

Fractional Brownian sheet

Functional limit theorems

Fractional Brownian motion

- Let $Z_H = \{Z_H(t) : t \in \mathbb{R}\}$ be a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$.
- That is, Z_H is a centered Gaussian process with covariance

$$\mathbf{E}[Z_H(t)Z_H(s)] = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \quad t,\, s \in \mathbb{R}.$$

- Specifically, we represent Z_H as $Z_H(t) = \int_{\mathbb{R}} G_H(t,u)\mathrm{d}W(u)$, $t \in \mathbb{R}$, where W is a standard Brownian motion and

$$G_H(t,u) := C(H)\Big((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\Big), \quad t,\, u \in \mathbb{R},$$

is the Mandelbrot–Van Ness kernel. Above, $x_+ := \max(x, 0)$ and $C(H) > 0$ is a constant that depends on H.

Hermite polynomials

- Let P_k, $k = 1, 2, \ldots$, be the Hermite polynomials given by

$$\begin{aligned} P_1(x) &= x, \\ P_2(x) &= x^2 - 1, \\ P_3(x) &= x^3 - 3x, \\ P_4(x) &= x^4 - 6x^2 + 3, \\ &\vdots \end{aligned}$$

- They are orthogonal polynomials with respect to the Gaussian measure on $\mathbb{R}$.

Hermite variations

- Let $k \geq 2$ and $n \in \mathbb{N}$.
- The k-th Hermite variation of the fBm Z_H on the grid $\{1/n : 0, 1, 2, \ldots, n\}$ is defined as

$$V_k^{(n)}(t) := \sum_{k=1}^{\lfloor nt \rfloor} P_k\Big(n^H\big(Z_H(\tfrac{i}{n}) - Z_H(\tfrac{i-1}{n})\big)\Big), \quad t \in [0,1].$$

- The realizations of the process $V_k^{(n)}$ belong to the Skorohod space $D([0,1])$.

Functional central limit theorem

- It follows from the classical results of Breuer and Major (1983) and Taqqu (1977) that if $H \in \left(0, 1 - \frac{1}{2k}\right)$, then

$$\left(Z_H,\ n^{-1/2} V_k^{(n)}\right) \xrightarrow[n\to\infty]{d} \left(Z_H,\ C'(H,k)B\right) \quad \text{in } D([0,1])^2,$$

where $C'(H,k) > 0$ is a constant and B is a standard Brownian motion, independent of Z_H.

- Moreover, in the critical case $H = 1 - \frac{1}{2k}$ it follows that

$$\left(Z_H,\ (n\log n)^{-1/2} V_k^{(n)}\right) \xrightarrow[n\to\infty]{d} \left(Z_H,\ C'(1-\tfrac{1}{2k},k)B\right)$$

in $D([0,1])^2$.

Non-central limit theorem

- When $H \in \left(1 - \frac{1}{2k}, 1\right)$, the Hermite variations have a limit, under suitable scaling, but the limit is non-Gaussian.
- More specifically, the result of Dobrushin and Major (1979) implies a non-central limit theorem: for any $t \in [0, 1]$,

$$n^{-(1-k(1-H))} V_k^{(n)}(t) \xrightarrow[n\to\infty]{L^2} C'(H,k) Y(t),$$

  where $\{Y(t) : t \in [0,1]\}$ is a k-th order Hermite process with Hurst parameter $1 - k(1-H) \in \left(\frac{1}{2}, 1\right)$.
- The Hermite process can be represented as a k-fold multiple Wiener integral with respect to Brownian motion.
- The second-order Hermite process is also known as the Rosenblatt process and its marginals are infinitely divisible.

Fractional Brownian sheet

- The fractional Brownian sheet (fBs), introduced by Ayache, Leger, and Pontier (2002), is an extension of the fBm to a multiparameter setting.
- It is defined by taking the "tensor product" of the correlation structures of multiple fBms with different Hurst parameters.
- More concretely, a d-parameter fBs with Hurst parameter $\mathbf{H} = (H_1, \ldots, H_d) \in (0,1)^d$ is a centered Gaussian process $\{Z_{\mathbf{H}}(\mathbf{t}) : \mathbf{t} \in \mathbb{R}^d\}$ with covariance

$$\mathbf{E}[Z_{\mathbf{H}}(\mathbf{t}) Z_{\mathbf{H}}(\mathbf{u})] = \prod_{\nu=1}^{d} \frac{1}{2}\left(|t_\nu|^{2H_\nu} + |u_\nu|^{2H_\nu} - |t_\nu - u_\nu|^{2H_\nu}\right),$$

for $\mathbf{t} = (t_1, \ldots, t_d)$, $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d$.

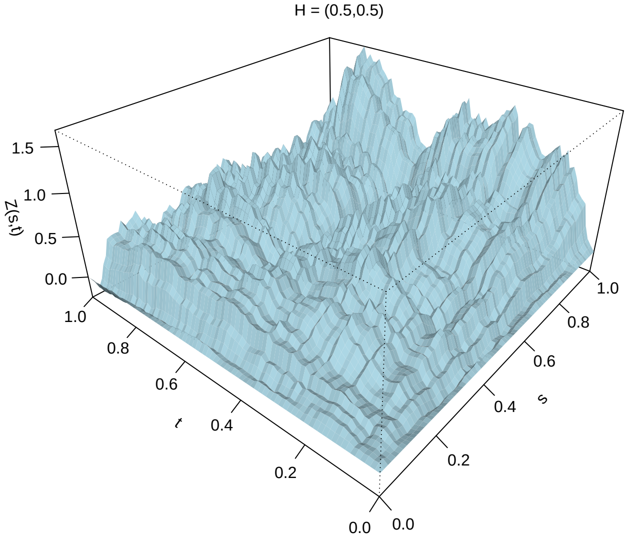
Fractional Brownian sheet (continued)

- The fBs is self-similar and has stationary increments (in the multiparameter sense). Moreover, it admits a continuous modification.
- But the smoothness properties of the realizations depend on the direction (anisotropy).
- Obviously, in the case $d = 1$ we recover the fBm.
- We will use the representation

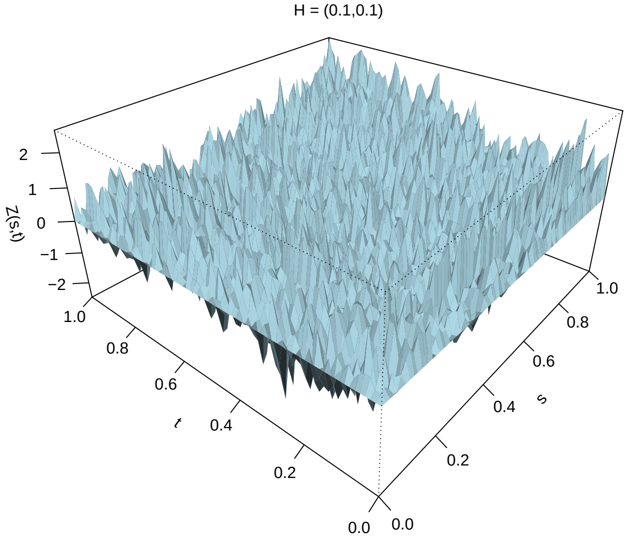
$$Z_{\mathbf{H}}(\mathbf{t}) = \int_{\mathbb{R}^d} \prod_{\nu=1}^{d} G_{H_\nu}(t_\nu, s_\nu) \mathcal{W}(\mathrm{d}s_1, \ldots, \mathrm{d}s_d), \quad \mathbf{t} \in \mathbb{R}^d,$$

where G_H is the Mandelbrot–Van Ness kernel and $\{\mathcal{W}(A) : A \in \mathcal{B}_b(\mathbb{R}^d)\}$ is a white noise on $\mathbb{R}^d$.

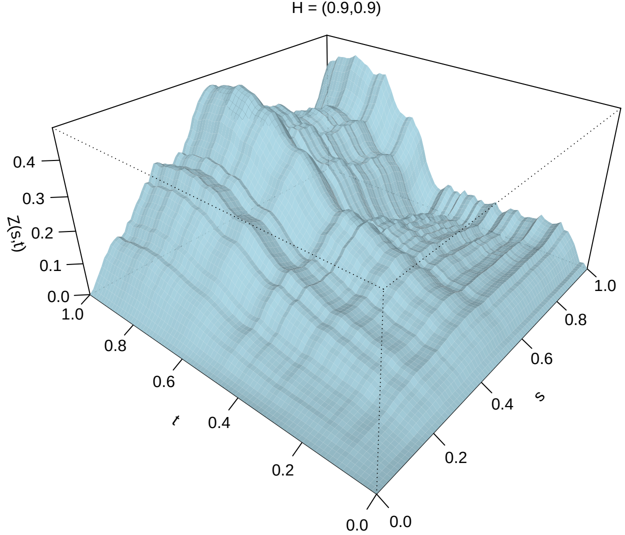
Simulation of the two-parameter fBs



Simulation of the two-parameter fBs



Simulation of the two-parameter fBs



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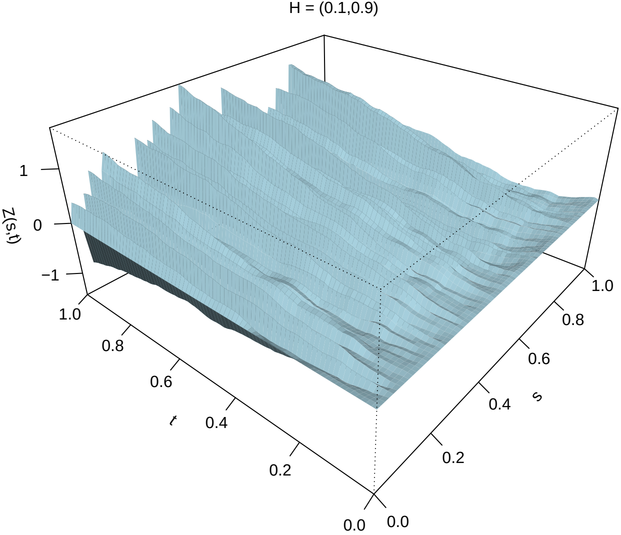
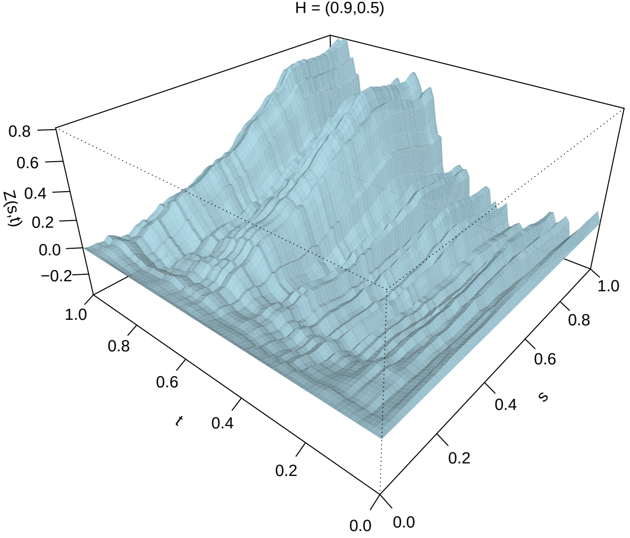


Simulation of the two-parameter fBs



Overview

- We want to extend the limit theorems for the fBm (as seen in the beginning) to the multiparameter setting with the fBs.
- Instead of Hermite variations, we consider more general functionals, generalized variations, where the Hermite polynomial is replaced with a more general function.
- When is the limit Gaussian?
    - Consider, e.g., the non-obvious case where H_1 is in the Gaussian regime and H_2 is in the non-Gaussian one.
- We extend the results of Réveillac, Stauch, and Tudor (2012).

Increments in the multiparameter setting

- Consider a function $h : \mathbb{R}^d \to \mathbb{R}$.
- The increment of h over a hyper-rectangle

$$[\mathbf{a}, \mathbf{b}) = [a_1, b_1) \times \cdots \times [a_d, b_d),$$

where $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^d$, is given by

$$h([\mathbf{a}, \mathbf{b})) := \sum_{\mathbf{i} \in \{0,1\}^d} (-1)^{d - \sum_{\nu=1}^{d} i_\nu} h\big((\mathbf{1} - \mathbf{i})\mathbf{a} + \mathbf{i}\mathbf{b}\big).$$

(Above, vectors are multiplied component-wise.)

- This definition can be recovered by differencing iteratively with respect to each of the arguments of the function h.

Generalized variations

- Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\mathbf{E}[f(\xi)^2] < \infty$ and $\mathbf{E}[f(\xi)] = 0$ with $\xi \sim N(0,1)$.
- Moreover, for $\mathbf{i} \in \mathbb{N}^d$ and $n \in \mathbb{N}$, denote

$$\square_{\mathbf{i}}^{(n)} := \left[\frac{i_1 - 1}{n}, \frac{i_1}{n}\right) \times \cdots \times \left[\frac{i_d - 1}{n}, \frac{i_d}{n}\right).$$

- We study the generalized variations of the fBs $Z_{\mathbf{H}}$,

$$U_f^{(n)}(\mathbf{t}) := \sum_{\mathbf{1} \leq \mathbf{i} \leq \lfloor n\mathbf{t} \rfloor} f\left(n^{\sum_{\nu=1}^{d} H_\nu} Z_{\mathbf{H}}\left(\square_{\mathbf{i}}^{(n)}\right)\right), \quad \mathbf{t} \in [0,1]^d,$$

  for $n \in \mathbb{N}$. (Above, all operations and relations involving vectors are understood component-wise.)
- The realizations of $U_f^{(n)}$ belong to the multiparameter Skorohod space $D([0,1]^d)$.

Hermite expansion

- The function f can be expanded in $L^2(\mathbb{R}, \gamma)$, where γ stands for the $N(0,1)$ distribution, using Hermite polynomials as

$$f(x) = \sum_{k=\underline{k}}^{\infty} a_k P_k(x),$$

where $a_{\underline{k}}, a_{\underline{k}+1}, \ldots$ are such that $a_{\underline{k}} \neq 0$ and $\sum_{k=\underline{k}}^{\infty} k! a_k^2 < \infty$.

- The index $\underline{k} \in \mathbb{N}$ is known as the Hermite rank of f.

Standing assumption

The coefficients $a_{\underline{k}}, a_{\underline{k}+1}, \ldots$ satisfy $\sum_{k=\underline{k}}^{\infty} 3^{\frac{k}{2}} \sqrt{k!} |a_k| < \infty$.

Rescaling

- The generalized variations need to be rescaled, in a way that depends on $\mathbf{H}$ and $\underline{k}$, to ensure convergence.
- To this end, we define for any $\nu = 1, \ldots, d$ and $n \in \mathbb{N}$,

$$\tau_\nu^{(n)} := \begin{cases} n^{-\frac{1}{2}}, & H_\nu \in \left(0, 1 - \frac{1}{2\underline{k}}\right), \\ (n \log n)^{-\frac{1}{2}}, & H_\nu = 1 - \frac{1}{2\underline{k}}, \\ n^{-(1-\underline{k}(1-H_\nu))}, & H_\nu \in \left(1 - \frac{1}{2\underline{k}}, 1\right). \end{cases}$$

- We define the rescaled variations by

$$\overline{U}_f^{(n)} := \left(\prod_{\nu=1}^{d} \tau_\nu^{(n)} \right) U_f^{(n)}, \quad n \in \mathbb{N}.$$

Functional central limit theorem

Theorem

Suppose that $\mathbf{H} \in (0,1)^d \setminus \left(1 - \frac{1}{2\underline{k}}, 1\right)^d$. *Then,*

$$\left(Z_{\mathbf{H}},\, \overline{U}_f^{(n)}\right) \xrightarrow[n\to\infty]{d} \left(Z_{\mathbf{H}},\, C''(\mathbf{H}, f)\widetilde{Z}_{\widetilde{\mathbf{H}}}\right) \quad \text{in } D([0,1]^d)^2,$$

where $C''(\mathbf{H}, f) > 0$ *is a constant and* $\widetilde{Z}_{\widetilde{\mathbf{H}}}$ *is a new fBs, independent of* $Z_{\mathbf{H}}$, *with Hurst parameter* $\widetilde{\mathbf{H}} \in [\frac{1}{2}, 1)^d$ *given by*

$$\widetilde{H}_\nu = \begin{cases} \frac{1}{2}, & H_\nu \in \left(0, 1 - \frac{1}{2\underline{k}}\right], \\ 1 - \underline{k}(1 - H_\nu), & H_\nu \in \left(1 - \frac{1}{2\underline{k}}, 1\right), \end{cases}$$

for any $\nu = 1, \ldots, d$.

Hermite sheet

- To describe the limit in the case $\mathbf{H} \in \left(1 - \frac{1}{2k}, 1\right)^d$, we need the so-called Hermite sheet.
- A k-th order d-parameter Hermite sheet with Hurst parameter $\mathbf{H} \in \left(\frac{1}{2}, 1\right)^d$ is a process $\{Y_{\mathbf{H}}(\mathbf{t}) : \mathbf{t} \in [0, \infty)^d\}$ given by

$$Y_{\mathbf{H}}(\mathbf{t}) := \int_{(\mathbb{R}^d)^k} J_{\mathbf{H}}^{(k)}\big(\mathbf{t}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}\big) \mathcal{W}(\mathbf{du}^{(1)}) \cdots \mathcal{W}(\mathbf{du}^{(k)}),$$

where $\mathcal{W}$ is the white noise on $\mathbb{R}^d$ and

$$J_{\mathbf{H}}^{(k)}\big(\mathbf{t}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}\big) := C'''(\mathbf{H}, k) \int_{[\mathbf{0}, \mathbf{t})} \prod_{\kappa=1}^{k} \prod_{\nu=1}^{d} \left(y_\nu - u_\nu^{(\kappa)}\right)_+^{-\frac{1}{2} - \frac{1 - H_\nu}{k}} \mathbf{dy}$$

for $\mathbf{t} \in [0, \infty)^d$ and $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)} \in \mathbb{R}^d$, with constant $C'''(\mathbf{H}, k) > 0$.

Hermite sheet (continued)

- This representation of the Hermite sheet is due to Clarke de la Cerda and Tudor (2014).
- The Hermite sheet has the same correlation structure (and self-similarity properties) as the fBs, but it is non-Gaussian whenever $k \geq 2$.
- In the case $k = 1$ it coincides with the fBs.
- In the case $d = 1$ it reduces to the Hermite process.

Functional non-central limit theorem

Theorem

Suppose that $\mathbf{H} \in \left(1 - \frac{1}{2\underline{k}}, 1\right)^d$. *Then,*

$$\overline{U}_f^{(n)} \xrightarrow[n\to\infty]{p} C''''(\mathbf{H}, f) Y_{\widetilde{\mathbf{H}}} \quad \text{in } D([0,1]^d),$$

where $C''''(\mathbf{H}, f) > 0$ *is a constant and* $Y_{\widetilde{\mathbf{H}}}$ *is a* $\underline{k}$*-th order Hermite sheet with Hurst parameter* $\widetilde{\mathbf{H}} \in (\frac{1}{2}, 1)^d$ *given by*

$$\widetilde{H}_\nu = 1 - \underline{k}(1 - H_\nu) \quad \text{for any } \nu = 1, \ldots, d.$$

Remark

The Hermite sheet $Y_{\widetilde{\mathbf{H}}}$ is driven by the same white noise $\mathcal{W}$ as the original fBs $Z_{\mathbf{H}}$.

References

- A. Ayache, S. Leger, and M. Pontier (2002): Drap brownien fractionnaire. *Potential Anal.* **17**(1), 893–898.
- P. Breuer and P. Major (1983): Central limit theorems for non-linear functionals of Gaussian fields. *J. Multivariate Anal.* **13**(3), 425–441.
- J. Clarke De la Cerda and C. A. Tudor (2014): Wiener integrals with respect to the Hermite random field and applications to the wave equation. *Collect. Math.* **65**(3), 341–356.
- R. L. Dobrushin and P. Major (1979): Non-central limit theorems for non-linear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete* **50**(1), 27–52.
- M. S. Pakkanen and A. Réveillac (2015+): Functional limit theorems for generalized variations of the fractional Brownian sheet. *Bernoulli*, to appear.
- A. Réveillac, M. Stauch, and C. A. Tudor (2012): Hermite variations of the fractional Brownian sheet. *Stoch. Dyn.* **12**(3), 1150021, 21 pp.
- M. S. Taqqu (1977): Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence. *Z. Wahrsch. Verw. Gebiete* **40**(3), 203–238.